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Quantum Optics
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Problem 1: Wigner function.

a) Analyse the Wigner function at the origin, i.e., $W(0,0) = \frac{1}{2\pi}\int dx \Psi^*(-\frac{x}{2})\Psi(\frac{x}{2})$. Construct the condition for $\Psi(x)$ and $\Psi^*(x)$ that produces the extremum value for $W(0,0)$ with the constraint $\int dx|\Psi(x)|^2 = 1$. Hint: use Lagrange multipliers.

b) Show that $-\frac{1}{\pi} \leq W(0,0) \leq \frac{1}{\pi}$. Why is this result valid for any phase-space point?

Problem 2: Casimir effect.

Consider two parallel perfectly conducting plates of dimensions L^2 separated by a distance $a \ll L$, both located inside a cubic conducting box of volume L^3.

a) Taking into account that at the surface of the perfectly conducting material the parallel component of the electric field must be zero, find the allowed wave numbers and the zero-point energy and show that the following expression for the energy is valid, where σ stands for the polarization:

$$\langle E \rangle = \frac{1}{2}\hbar c \sum_{\sigma} \sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} \sum_{n_z=0}^{\infty} \sqrt{\left(\frac{n_x\pi}{L}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2 + \left(\frac{n_z\pi}{a}\right)^2} \quad .$$

In the Coulomb gauge, the transversality condition determines one of the amplitude vector components in terms of the other two, unless $n_i = 0$ for any i. Apart from this exception, the modes are characterized by a two-dimensional amplitude vector, that is by two polarizations. In the exceptional case of $n_x = 0$ or $n_y = 0$ or $n_z = 0$, the electric field is uniform in a given direction, and the modes have only one polarization [1]. This means that for the $n_i \neq 0$ terms we can sum over polarizations to get a factor 2.

b) Observing that the x and y terms are added in quadrature and that $L \gg a$, we can take the limit of these sums as integrals and make a change to polar coordinates. It is even clearer now that $\langle E \rangle$ is infinite. Quantizing the electromagnetic field at each point in space must yield an infinite energy! However, we are free to set the zero of a potential energy wherever we like, even

[1] When $k_i = 0$ the modes of the electric field propagate parallel to the surface perpendicular to $\mathbf{e}_i$ (the versor along the i-axis). The electric field polarization must be perpendicular to the propagation direction. $\mathbf{E}$ must also have zero tangential components on the surface $\perp \mathbf{e}_i$ due to the perfectly conducting character of the boundaries. The field is thus homogeneous along the i-axis, which is the only allowed polarization direction (this actually means that there is an associated surface charge with such a mode, but this is not forbidden). For $k_i \neq 0$, the field vanishes at the plates anyway, so there is no restriction on the polarization directions apart from transversality.

at infinity. To that end, we consider the difference between the obtained zero-point energy at two different configurations, one with $a \ll L$ and another one with $a \lesssim L$ (you should rescale the latter just like what was done in the lectures). Show that the energy difference $\langle \delta E \rangle$ is porportional to a^{-3} times a divergent sum of integrals.

c) Considering the fact that each conductor has a plasma frequency ω_p, which is the minimum oscillation frequency the electrons in the conductor can support, the conducting plates are supposed to be transparent to photons of frequencies above ω_p due to the plates' electrons oscillating in resonance with the high frequency waves. Because of this, we multiply the contributions to the total energy of each mode k by a regulator function $f(k/k_m)$, which is unity for $k < k_m$, approaches 0 at infinity, and is 1/2 at $k = k_m$. The exact value of k_m and the shape of f may be phenomenologically obtained, but will not be especially important in this case, since the answer is independent of the regularization parameters. Show that the corrected formula for the energy difference is given by:

$$\langle \delta E \rangle = L^2 \hbar c \frac{\pi^2}{4a^3} \left[\sum_{n_z=0}^{\infty} (1 - \delta_{0,n_z}/2) \int_{n_z^2}^{\infty} \sqrt{\gamma} f\left(\frac{\sqrt{\gamma}\pi}{a k_m}\right) d\gamma - \int_0^{\infty} \int_{n_z^2}^{\infty} \sqrt{\gamma} f\left(\frac{\sqrt{\gamma}\pi}{a k_m}\right) d\gamma dn_z \right]$$

d) Consider the Euler-Maclaurin formula:

$$\int_0^N f(x) dx - \sum_{n=0}^{N} (1 - \delta_{0,n_z}/2) f(n) = \sum_{k=2}^{p} \frac{B_k}{k!} (f^{(k-1)}(N) - f^{(k-1)}(0)) + R$$

where R is an error term that decreases as N increases, B_k are the Bernoulli numbers ($-1/2, 1/6, 0, -1/30$ for $k = 1, 2, 3, 4$) and p is the degree of differentiability of f (in this example you should go up to the first non-zero term only). Identifying $f(n)$ with the inner integral in both terms of the expression for $\langle \delta E \rangle$ and applying the fundamental theorem of calculus, find its derivatives and apply the Euler-Maclaurin series to find the final expression for the energy difference (don't forget to use the properties of the function f).

e) Considering that $dV = L^2 da$ for a slowly and adiabatically varying distance between the plates, find the Casimir pressure with the help of the expression $P = -\partial E/\partial V$.